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New two-mode coherent-entangled state and its application

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Abstract

We introduce the so-called coherent-entangled state in the two-mode Fock space; this state exhibits both the properties of the coherent and entangled states. Generalized $\mathcal{P}$-representation in the coherent-entangled state is constructed. A protocol for producing the ideal coherent-entangled state is presented. It is shown that the bipartite entangled state can be a superposition of the coherent-entangled states.

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1. Introduction

Various coherent states have been widely used in quantum mechanics and quantum optics since Glauber [1] and Klauder and Skagerstam [2] initially introduced the coherent state of the harmonic oscillator. Formally, the bosonic coherent state $|z\rangle_i = \exp(za_i^\dagger)|0\rangle_i$ is the eigenket of the Bose annihilation operator $a_i|z\rangle_i = z|z\rangle_i$. $|z\rangle_i$ is non-orthogonal, i.e., ${}_i\langle z'|z\rangle_i = \exp(zz'^*)$. Bhaumik *et al* [3] defined the so-called conserved-charge coherent state $|q,\alpha\rangle$, which is the common eigenvector of two-mode annihilation operators a_1a_2 and the charge operator $Q \equiv a_1^\dagger a_1 - a_2^\dagger a_2$,

$$Q|q,\alpha\rangle = q|q,\alpha\rangle, \qquad a_1a_2|q,\alpha\rangle = \alpha|q,\alpha\rangle, \tag{1}$$

and

$$|q,\alpha\rangle = c_q\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!(n+q)!}}|n+q,n\rangle, \tag{2}$$

where c_q is a normalization factor, and $|n+q,n\rangle$ is the two-mode number state. $|q,\alpha\rangle$ is also called a pair coherent state by Agarwal and Tara [4]. Subsequently, the common eigenstates $|q,\lambda\rangle$ of Q and $G \equiv (a_1+a_2^\dagger)(a_1^\dagger+a_2)$ were also explored [5],

$$Q|q,\lambda\rangle = q|q,\lambda\rangle, \qquad G|q,\lambda\rangle = \lambda|q,\lambda\rangle, \qquad \lambda \geqslant 0, \tag{3}$$

since $[Q, G] = 0$. The explicit form of $|q, \lambda\rangle$ in two-mode Fock space is

$$|q, \lambda\rangle = \mathrm{e}^{-\lambda/2} \sum_{n=\max(0,-q)}^{\infty} H_{n+q,n}(\sqrt{\lambda}, \sqrt{\lambda}) \frac{1}{\sqrt{(n+q)!n!}} |n+q, n\rangle, \tag{4}$$

where $H_{m,n}$ is the two-variable Hermite polynomial, defined as

$$H_{m,n}(z, z^*) = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k m!n!}{k!(m-k)!(n-k)!} z^{m-k} z^{*n-k}. \tag{5}$$

In this work, we will construct a new kind of two-mode coherent state, which we call the coherent-entangled state $|\alpha, x\rangle$, so named as it is the common eigenvector of $(X_1 + X_2)/2$ (the centre-of-mass coordinate of two particles) and $a_1 - a_2$, i.e.,

$$\frac{1}{2}(X_1 + X_2)|\alpha, x\rangle = \frac{x}{\sqrt{2}}|\alpha, x\rangle, \qquad (a_1 - a_2)|\alpha, x\rangle = \alpha|\alpha, x\rangle, \tag{6}$$

since

$$[\tfrac{1}{2}(X_1 + X_2), a_1 - a_2] = 0, \tag{7}$$

where $X_i = (a_i + a_i^\dagger)/\sqrt{2}$ is the coordinate operator. The motivation to search for $|\alpha, x\rangle$ lies in the fact that, as one can see shortly later, this state exhibits not only the property of coherent state, e.g., $|\alpha, x\rangle$ span an overcomplete and partly orthonormal set, but also is an entangled state. Quantum entanglement was first proposed by Einstein, Podolsky and Rosen (EPR) [6] in challenging the incompleteness of quantum mechanics. We can consider $|\alpha, x\rangle$ a new quantum mechanical representation which will bring convenience in dealing with some problems in quantum optics. Our paper is organized as follows: in section 2, we present the explicit form of $|\alpha, x\rangle$ in two-mode Fock space and discuss its main properties. In section 3, we present a protocol for producing the ideal $|\alpha, x\rangle$ state. In section 4, we briefly discuss some applications of $|\alpha, x\rangle$ in quantum optics theory. Section 5 is devoted to investigating the Schmidt decomposition and the relation between $|\alpha, x\rangle$ and the EPR entangled state, i.e., the bipartite entangled state can be a superposition of the coherent-entangled states.

2. Properties and the explicit form of $|\alpha, x\rangle$

After some tries, we find that the explicit form of $|\alpha, x\rangle$ in two-mode Fock space is

$$|\alpha, x\rangle = \exp(-\tfrac{1}{4}|\alpha|^2 - \tfrac{1}{2}x^2) \exp[-\tfrac{1}{4}(a_1^\dagger + a_2^\dagger)^2 + (x + \tfrac{1}{2}\alpha)a_1^\dagger + (x - \tfrac{1}{2}\alpha)a_2^\dagger]|00\rangle, \tag{8}$$

where $\alpha = \alpha_1 + \mathrm{i}\alpha_2$, $\mathrm{i}^2 = -1$, $|00\rangle$ is annihilated by a_i. In fact, using $[a_i, a_j^\dagger] = \delta_{ij}$, we know

$$a_1|\alpha, x\rangle = [\tfrac{1}{2}(\alpha - a_1^\dagger - a_2^\dagger) + x]|\alpha, x\rangle, \qquad a_2|\alpha, x\rangle = [-\tfrac{1}{2}(a_2^\dagger + a_1^\dagger + \alpha) + x]|\alpha, x\rangle. \tag{9}$$

It follows that

$$(a_1 - a_2)|\alpha, x\rangle = \alpha|\alpha, x\rangle \tag{10}$$

and

$$(a_1 + a_2)|\alpha, x\rangle = (-a_1^\dagger - a_2^\dagger + 2x)|\alpha, x\rangle. \tag{11}$$

Then, according to (11), we have

$$\frac{1}{2}(X_1 + X_2)|\alpha, x\rangle = \frac{1}{\sqrt{2}} x|\alpha, x\rangle. \tag{12}$$

So we have proved that $|\alpha, x\rangle$ is really the eigenvector of both $(a_1 - a_2)$ and $(X_1 + X_2)/2$.

We shall then check if $|\alpha, x\rangle$ possesses the completeness relation and the orthogonal relation. Using the normal ordered product of the two-mode vacuum projector

$$|00\rangle\langle 00| = :\exp[-a_1^\dagger a_1 - a_2^\dagger a_2]: \tag{13}$$

and the technique of integral within an ordered product (IWOP) of operators [7, 8] (see [9] for a recent review), we can smoothly prove the completeness relation of $|\alpha, x\rangle$ (let $\mathrm{d}^2\alpha = \mathrm{d}\alpha_1 \mathrm{d}\alpha_2$),

$$\begin{aligned}
&\int \frac{\mathrm{d}x}{\sqrt{\pi}} \int \frac{\mathrm{d}^2\alpha}{2\pi} |\alpha, x\rangle\langle\alpha, x| \\
&= \int \frac{\mathrm{d}x}{\sqrt{\pi}} \int \frac{\mathrm{d}^2\alpha}{2\pi} :\exp\left[-\frac{1}{2}|\alpha|^2 - x^2 - \frac{1}{4}(a_1^\dagger + a_2^\dagger)^2 + \left(x + \frac{1}{2}\alpha\right)a_1^\dagger + \left(x - \frac{1}{2}\alpha\right)a_2^\dagger \right. \\
&\quad \left. - a_1^\dagger a_1 - a_2^\dagger a_2 - \frac{1}{4}(a_1 + a_2)^2 + \left(\frac{1}{2}\alpha^* + x\right)a_1 + \left(x - \frac{1}{2}\alpha^*\right)a_2\right] := 1.
\end{aligned} \tag{14}$$

To calculate the overlap $\langle\alpha', x'|\alpha, x\rangle$, we employ the over-completeness relation of the two-mode coherent state [10]

$$\int \frac{\mathrm{d}z_1 \mathrm{d}z_2}{\pi^2} |z_1, z_2\rangle\langle z_1, z_2| = 1 \tag{15}$$

and notice the overlap

$$\begin{aligned}
\langle z_1, z_2|\alpha, x\rangle = {} & \exp\left(-\tfrac{1}{4}|\alpha|^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}|z_1|^2 - \tfrac{1}{2}|z_2|^2\right) \\
& \times \exp\left[-\tfrac{1}{4}(z_1^* + z_2^*)^2 + \left(x + \tfrac{1}{2}\alpha\right)z_1^* + \left(x - \tfrac{1}{2}\alpha\right)z_2^*\right].
\end{aligned} \tag{16}$$

We obtain

$$\begin{aligned}
\langle\alpha', x'|\alpha, x\rangle &= \int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2} \langle\alpha', x'|z_1, z_2\rangle\langle z_1, z_2|\alpha, x\rangle \\
&= D_1 \int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2} \exp\left\{ -\sum_{i=1}^{2}\left(|z_i|^2 + \frac{1}{4}z_i^2 + \frac{1}{4}z_i^{*2}\right) + \left(x' + \frac{1}{2}\alpha'^* - \frac{1}{2}z_2\right)z_1 \right. \\
&\quad \left. + \left(x + \frac{1}{2}\alpha - \frac{1}{2}z_2^*\right)z_1^* + \left(x' - \frac{1}{2}\alpha'^*\right)z_2 + \left(x - \frac{1}{2}\alpha\right)z_2^* \right\},
\end{aligned} \tag{17}$$

where

$$D_1 = \exp\left[-\tfrac{1}{4}(|\alpha|^2 + |\alpha'|^2) - \tfrac{1}{2}(x^2 + x'^2)\right]. \tag{18}$$

Using the mathematical formula

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp(\zeta|z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}) = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left[\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right], \tag{19}$$

its convergence condition is

$$\operatorname{Re}(\zeta + f + g) < 0, \qquad \operatorname{Re}\left(\frac{\zeta^2 - 4fg}{\xi + f + g}\right) < 0, \tag{20}$$

or

$$\operatorname{Re}(\zeta - f - g) < 0, \qquad \operatorname{Re}\left(\frac{\zeta^2 - 4fg}{\xi - f - g}\right) < 0, \tag{21}$$

to perform the integral over $\mathrm{d}^2 z_1$ in (17), we obtain

$$\langle\alpha', x'|\alpha, x\rangle = 2D_2 \int \frac{\mathrm{d}^2 z_2}{\sqrt{3}\pi} \exp\left[\frac{1}{3}(-2|z_2|^2 - z_2^2 - z_2^{*2} + Az_2 + Bz_2^*)\right], \tag{22}$$

where

$$A = 4x - 2x' - \alpha^* - \alpha', \qquad B = 4x' - 2x - \alpha^* - \alpha' \tag{23}$$

and

$$\begin{aligned} D_2 = {} & \exp(-\tfrac{1}{4}|\alpha|^2 - \tfrac{1}{4}|\alpha'|^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}x'^2) \\ & \times \exp[-\tfrac{1}{3}(\tfrac{1}{2}\alpha'^* + x')^2 - \tfrac{1}{3}(\tfrac{1}{2}\alpha + x)^2 + \tfrac{4}{3}(\tfrac{1}{2}\alpha + x)(\tfrac{1}{2}\alpha'^* + x')]. \end{aligned} \tag{24}$$

At this point, we notice that, when one attempts to use formula (19) to further carry out the integral over $\mathrm{d}^2 z_2$ in (22), one encounters

$$\frac{1}{\sqrt{\zeta^2 - 4fg}} = 1 \Big/ \sqrt{\left(\tfrac{2}{3}\right)^2 - 4\tfrac{1}{3}\cdot\tfrac{1}{3}},$$

which shows a singularity. To overcome this difficulty, we introduce a limiting process, i.e. we rewrite (22) as

$$\begin{aligned} \langle\alpha', x'|\alpha, x\rangle &= 2D_2 \lim_{t\to 1} \int \frac{\mathrm{d}^2 z_2}{\pi} \exp\left[\frac{1}{3}(-2|z_2|^2 - tz_2^2 - tz_2^{*2} + Az_2 + Bz_2^*)\right] \\ &= \sqrt{3} D_2 \exp\left[\frac{1}{24}(A^2 + B^2)\right] \lim_{t\to 1} \frac{1}{\sqrt{1-t^2}} \exp\left[\frac{-\sqrt{3}}{1-t^2}(x - x')^2\right]. \end{aligned} \tag{25}$$

Then, using the limiting form of Dirac's delta function

$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\sqrt{\pi\epsilon}} \exp\left(\frac{-x^2}{\epsilon}\right), \tag{26}$$

we finally get

$$\langle\alpha', x'|\alpha, x\rangle = \sqrt{\pi} \exp[-\tfrac{1}{4}(|\alpha|^2 + |\alpha'|^2) + \tfrac{1}{2}\alpha\alpha'^*]\delta(x - x'). \tag{27}$$

Particularly, when $\alpha = \alpha'$,

$$\langle\alpha, x|\alpha', x'\rangle = \sqrt{\pi}\delta(x - x'). \tag{28}$$

Thus the coherent-entangled state $|\alpha, x\rangle$ is only partly orthogonal. From (14) and (27), we conclude that $|\alpha, x\rangle$ exhibits the properties of both the coherent and entangled states.

3. Generating the state $|\alpha, x\rangle$ by beam splitters

Furusawa *et al* [11] have pointed out that beam splitters can play a role in producing quantum entanglement, and we have shown in [12] that the ideal single-mode squeezed state $\exp(-a_1^{\dagger 2}/2)|0\rangle_1$ [13] and the vacuum state $|0\rangle_2$ overlapping on a beam splitter may produce the generalized two-mode squeezed state $\exp[-\frac{1}{4}(a_1^\dagger + a_2^\dagger)^2]|00\rangle$ at the output. In fact, the role of a beam splitter operator [14–17] is expressed by

$$B = \exp\left[-\frac{\theta}{2}(a_1^\dagger a_2 - a_1 a_2^\dagger)\right]. \tag{29}$$

It operates on the input state $\exp(-a_1^{\dagger 2}/2)|0\rangle_1 \otimes |0\rangle_2$ and yields

$$B\exp(-a_1^{\dagger 2}/2)|0\rangle_1 \otimes |0\rangle_2 = \exp[-(a_2^\dagger\cos\theta + a_1^\dagger\sin\theta)^2]|00\rangle. \tag{30}$$

When $\theta = \pi/4$, a symmetric beam splitter case, (30) becomes

$$\exp[-\tfrac{1}{4}(a_1^\dagger + a_2^\dagger)^2]|00\rangle. \tag{31}$$

Then, we operate two local oscillator laser beams, denoted by two displacement operators,

$$\begin{aligned} D_1[\tfrac{1}{2}(x+\alpha)] &= \exp[\tfrac{1}{2}(x+\alpha)a_1^\dagger - \tfrac{1}{2}(x+\alpha^*)a_1], \\ D_2[\tfrac{1}{2}(x-\alpha)] &= \exp[\tfrac{1}{2}(x-\alpha)a_2^\dagger - \tfrac{1}{2}(x-\alpha^*)a_2], \end{aligned} \tag{32}$$

on the output state $\exp[-\frac{1}{4}(a_1^\dagger + a_2^\dagger)^2]|00\rangle$, and the result is

$$\begin{aligned} &D_1 D_2 \exp[-\tfrac{1}{4}(a_1^\dagger + a_2^\dagger)^2]|00\rangle \\ &\quad = \exp[-\tfrac{1}{4}[a_1^\dagger - \tfrac{1}{2}(x+\alpha^*) + a_2^\dagger - \tfrac{1}{2}(x-\alpha^*)]^2]D_1|0\rangle_1 \otimes D_2|0\rangle_2 = |\alpha, x\rangle. \end{aligned} \tag{33}$$

Thus the ideal coherent-entangled state can be implemented. One can achieve the displacement $D_1[\frac{1}{2}(x+\alpha)]$ experimentally by reflecting the light field from an almost perfect reflecting mirror and adding through the mirror a field (a strong coherent local oscillator) that has been phase- and amplitude-modulated according to the values $\frac{1}{2}(x+\alpha)$, and a similar analysis is true for $D_2[\frac{1}{2}(x-\alpha)]$.

4. Generalized $\mathcal{P}$-representation in coherent-entangled state

Since $|\alpha, x\rangle$ also possesses properties of a coherent state, it may therefore be used as a basis set to expand density operator $\rho[a_1^\dagger - a_2^\dagger, a_1 - a_2]$ to establish the so-called generalized $\mathcal{P}$-representation [1, 12, 18]

$$\rho[a_1^\dagger - a_2^\dagger, a_1 - a_2] = \int \frac{\mathrm{d}x}{\sqrt{\pi}} \int \frac{\mathrm{d}^2\alpha}{2\pi} P(\alpha, \alpha^*)|\alpha, x\rangle\langle\alpha, x|. \tag{34}$$

Multiplying the left- and right-hand sides of (34) with $\langle -\alpha', -x'|$ and $|\alpha', x'\rangle$, respectively, and using (27), we have

$$
\begin{aligned}
&\langle -\alpha', -x'|\rho[a_1^\dagger - a_2^\dagger, a_1 - a_2]|\alpha', x'\rangle \\
&\quad = \int \frac{\mathrm{d}x}{\sqrt{\pi}} \int \frac{\mathrm{d}^2\alpha}{2\pi} P(\alpha,\alpha^*) \langle -\alpha', -x'|\alpha, x\rangle \langle \alpha, x|\alpha', x'\rangle \\
&\quad = \int \frac{\mathrm{d}x}{\sqrt{\pi}} \delta(x - x')\delta(x + x') \int \frac{\mathrm{d}^2\alpha}{2} P(\alpha,\alpha^*) \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\alpha'|^2 + \frac{1}{2}(\alpha^*\alpha' - \alpha\alpha'^*)\right] \\
&\quad = \delta(2x') \int \frac{\mathrm{d}^2\alpha}{2\sqrt{\pi}} P(\alpha,\alpha^*) \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\alpha'|^2 + \frac{1}{2}(\alpha^*\alpha' - \alpha\alpha'^*)\right].
\end{aligned} \tag{35}
$$

Since $\frac{1}{2}(\alpha^*\alpha' - \alpha\alpha'^*)$ is a pure imaginary number, we can take the Fourier inverse of (35) to obtain

$$
\begin{aligned}
P(\alpha,\alpha^*) = {}& \frac{1}{2\pi^{3/2}} \exp\left(\frac{1}{2}|\alpha|^2\right) \int \mathrm{d}^2\alpha' \exp\left[\frac{1}{2}|\alpha'|^2 + \frac{1}{2}(\alpha\alpha'^* - \alpha^*\alpha')\right] \\
& \times \int \mathrm{d}x' \langle -\alpha', -x'|\rho[a_1^\dagger - a_2^\dagger, a_1 - a_2]|\alpha', x'\rangle,
\end{aligned} \tag{36}
$$

which is a generalization by Mehta [19].

Moreover, $|\alpha, x\rangle$ is very convenient to derive some operator identities. First let us consider one such operator expression,

$$
O \equiv \exp[f(a_1 - a_2)^2] \exp[g(a_1^\dagger - a_2^\dagger)^2]. \tag{37}
$$

By inserting the completeness relation of $|\alpha, x\rangle$ (14) into (37) and using the IWOP technique, we obtain

$$
\begin{aligned}
O = {}& \int_{-\infty}^{\infty} \frac{\mathrm{d}x\,\mathrm{d}^2\alpha}{\sqrt[3]{\pi}} \exp[f(a_1 - a_2)^2]|\alpha, x\rangle\langle\alpha, x| \exp[g(a_1^\dagger - a_2^\dagger)^2] \\
= {}& \int_{-\infty}^{\infty} \frac{\mathrm{d}x\,\mathrm{d}^2\alpha}{2\sqrt[3]{\pi}} : \exp\left[-\tfrac{1}{2}|\alpha|^2 + f\alpha^2 + g\alpha^{*2} + \left(x + \tfrac{1}{2}\alpha\right) a_1^\dagger + \left(x - \tfrac{1}{2}\alpha\right) a_2^\dagger \right. \\
& \left. + \left(x + \tfrac{1}{2}\alpha^*\right) a_1 + \left(x - \tfrac{1}{2}\alpha^*\right) a_2 \right] \\
& \times \exp\left[-\tfrac{1}{4}(a_1^\dagger + a_2^\dagger)^2 - \tfrac{1}{4}(a_1 + a_2)^2 - a_1^\dagger a_1 - a_2^\dagger a_2 - x^2\right] : \\
= {}& \frac{1}{\sqrt{1 - 16fg}} \exp\left[\frac{g(a_1^\dagger - a_2^\dagger)^2}{1 - 16fg}\right] : \exp\left[\frac{1}{2}\left(\frac{1}{1 - 16fg} - 1\right)(a_1^\dagger - a_2^\dagger)(a_1 - a_2)\right] : \\
& \times \exp\left[\frac{f(a_1 - a_2)^2}{1 - 16fg}\right] \\
= {}& \frac{1}{\sqrt{1 - 16fg}} \exp\left[\frac{g(a_1^\dagger - a_2^\dagger)^2}{1 - 16fg}\right] \\
& \times \exp\left[\frac{1}{2}\ln\left(\frac{1}{1 - 16fg}\right)(a_1^\dagger - a_2^\dagger)(a_1 - a_2)\right] \exp\left[\frac{f(a_1 - a_2)^2}{1 - 16fg}\right].
\end{aligned} \tag{38}
$$

In a similar manner, we can derive the following operator entity:

$$\exp[f(a_1-a_2)^2]\exp[g(X_1+X_2)^2]$$
$$=\int_{-\infty}^{\infty}\frac{\mathrm{d}x\,\mathrm{d}^2\alpha}{2\sqrt[3]{\pi}}\exp[f\alpha^2]|\alpha,x\rangle\langle\alpha,x|\exp[2gx^2]$$
$$=\frac{1}{\sqrt{1-2g}}\exp\left[\frac{g(a_1^\dagger+a_2^\dagger)^2}{2(1-2g)}\right]\exp\left[\frac{1}{2}\ln\left(\frac{1}{1-2g}\right)(a_1^\dagger+a_2^\dagger)(a_1+a_2)\right]$$
$$\times\exp\left[\left(\frac{g}{2(1-2g)}+f\right)(a_1-a_2)^2\right]. \tag{39}$$

Especially, when $f=0$,

$$\exp[g(X_1+X_2)^2]=\frac{1}{\sqrt{1-2g}}\exp\left[\frac{g(a_1^\dagger+a_2^\dagger)^2}{2(1-2g)}\right]$$
$$\times\exp\left[\frac{1}{2}(a_1^\dagger+a_2^\dagger)\ln\left(\frac{1}{1-2g}\right)(a_1+a_2)\right]\exp\left[\frac{g(a_1-a_2)^2}{2(1-2g)}\right]. \tag{40}$$

Thus we know

$$\exp[g(X_1+X_2)^2]|00\rangle=\frac{1}{\sqrt{1-2g}}\exp\left[\frac{g(a_1^\dagger+a_2^\dagger)^2}{2(1-2g)}\right]|00\rangle, \tag{41}$$

which is a generalized two-mode squeezed state. This kind of state can be the output state of a beam splitter when the two input states are a one-mode squeezed state and a vacuum state [12].

5. Generating EPR entangled state by superposition of $|\alpha, x\rangle$

Furthermore, by making the Fourier transform

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_2}{2\pi}|\alpha=\alpha_1+\mathrm{i}\alpha_2,x\rangle\exp[-\mathrm{i}u\alpha_2]=\exp\left(-\frac{1}{8}\alpha_1^2-\frac{1}{2}u^2+\frac{1}{2}u\alpha_1\right)$$
$$\times\left|q_1=\frac{x}{\sqrt{2}}+\frac{u+(\alpha_1/2)}{\sqrt{2}}\right\rangle\otimes\left|q_2=\frac{x}{\sqrt{2}}-\frac{u+(\alpha_1/2)}{\sqrt{2}}\right\rangle, \tag{42}$$

where

$$|q_i\rangle=\frac{1}{\pi^{1/4}}\exp\left[-\frac{q_i^2}{2}+\sqrt{2}q_ia_i^\dagger-\frac{a_i^{\dagger 2}}{2}\right]|0\rangle \tag{43}$$

is the coordinate eigenstate and then taking the inverse Fourier transform of (42), we obtain

$$|\alpha=\alpha_1+\mathrm{i}\alpha_2,x\rangle=\exp\left(-\frac{1}{8}\alpha_1^2\right)\int_{-\infty}^{\infty}\mathrm{d}u\exp[\tfrac{1}{2}u(\alpha_1+2\mathrm{i}\alpha_2-u)]$$
$$\times\left|q_1=\frac{x}{\sqrt{2}}+\frac{u+\frac{1}{2}\alpha_1}{\sqrt{2}}\right\rangle\otimes\left|q_2=\frac{x}{\sqrt{2}}-\frac{u+\frac{1}{2}\alpha_1}{\sqrt{2}}\right\rangle, \tag{44}$$

which is the Schmidt decomposition of $|\alpha, x\rangle$ and confirms that $|\alpha, x\rangle$ itself is an entangled state.

On the other hand, by performing the integral over $\mathrm{d}\alpha_1$ for $|\alpha, x\rangle$, we have

$$\int_{-\infty}^{\infty} \mathrm{d}\alpha_1 |\alpha = \alpha_1 + \mathrm{i}\alpha_2, x\rangle = 2\sqrt{\pi} \exp[-\tfrac{1}{4}\alpha_2^2 - \tfrac{1}{2}x^2 + a_1^\dagger(x + \tfrac{1}{2}\mathrm{i}\alpha_2) + a_2^\dagger(x - \tfrac{1}{2}\mathrm{i}\alpha_2) - a_1^\dagger a_2^\dagger]|00\rangle. \tag{45}$$

Comparing (45) with the standard expression of the EPR entangled state [20–22]

$$|\xi\rangle = \exp[-\tfrac{1}{2}|\xi|^2 + \xi a_1^\dagger + \xi^* a_2^\dagger - a_1^\dagger a_2^\dagger]|00\rangle, \qquad \xi = \xi_1 + \mathrm{i}\xi_2, \tag{46}$$

which is the common eigenvector of the two particles' centre-of-mass coordinate and the relative momentum, i.e.

$$(X_1 + X_2)|\xi\rangle = \sqrt{2}\xi_1|\xi\rangle, \qquad (P_1 - P_2)|\xi\rangle = \sqrt{2}\xi_2|\xi\rangle, \tag{47}$$

where $P_i = (a_i - a_i^\dagger)/(\mathrm{i}\sqrt{2})$, we have

$$\int_{-\infty}^{\infty} \mathrm{d}\alpha_1 |\alpha = \alpha_1 + \mathrm{i}\alpha_2, x\rangle = 2\sqrt{\pi} \exp(-\tfrac{1}{8}\alpha_2^2)|\xi = x + \tfrac{1}{2}\mathrm{i}\alpha_2\rangle. \tag{48}$$

Equation (48) shows that the superposition of the states $|\alpha = \alpha_1 + \mathrm{i}\alpha_2, x\rangle$ along the α_1 axis yields the EPR entangled state, which provides new insights into the bipartite entangled state, since it indicates how the bipartite entangled state can be a superposition of the coherent-entangled states. In the last two decades, superposition of states in quantum optics has drawn the attention of many physicists, because this can produce nonclassical states, e.g. it is pointed out in [23] that a superposition of two coherent states is useful in producing nonclassical states.

Finally, we mention the work of Sanders *et al* [24], which also uses the terminology 'entangled coherent-state', but its concept is completely different from ours. Actually, the entangled coherent-state in [24] is constructed by operating a two-mode squeezing operator on a state (the n-photon state in one mode and no photons in the other mode), which is expressed as a superposition of coherent states over a circle in the complex phase space.

6. Summary

In conclusion, with the aid of the IWOP technique, we have proposed a new state representation $|\alpha, x\rangle$, which we call the coherent-entangled state. We proved the completeness relation of $|\alpha, x\rangle$ and showed that $|\alpha, x\rangle$ is only partly orthogonal. A feasible experimental set-up to generate the state $|\alpha, x\rangle$ using beamsplitters was also proposed. A generalized $\mathcal{P}$-representation was constructed in the coherent-entangled state. Finally, it is revealed that superposition of the coherent-entangled states can produce the EPR entangled state.

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